

HOM-LIE ALGEBROIDS

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ABSTRACT. We define hom-Lie algebroids, a definition that may seem cumbersome at first, but which is justified, first, by a one-to-one correspondence with hom-Gerstenhaber algebras, a notion that we also introduce, and several examples, including hom-Poisson structures.

1. INTRODUCTION

There is an increasing interest for hom-structures, which are, to make a long story short, Lie algebra-like structures, equipped with an additional map, say α , and in which the cubic identities (e.g. the associativity or the Jacobi conditions) are replaced by a relation of the same form, in which the first variable, say x , is replaced by $\alpha(x)$, so that the henceforth obtained condition is always true when restricted to the kernel of that map. Originally, the study of hom-Lie algebras, initiated by [HLS06] showed a natural occurrence of this notion while studying cocycles of the Virasoro algebra. In the following years, Makhlouf, Silvestrov and their coauthors [MS1] have showed that several classical algebraic structures admit natural generalizations when, instead of just a vector space, we start with a vector space and an automorphism of it, leading to investigate hom-associative algebras [MS1], hom-Jordan algebras [MS4], admissible algebras [MS2], hom-Poisson algebras [MS3], to cite a few. The interest in hom-Poisson structure is likely to grow again due to the recent thesis of Olivier Elchinger [E], who introduces quantization of Hom-Poisson structures, giving, in particular, explicit formulas for the Moyal product, since this raises the question of integration of hom-Lie algebroids.

Our purpose is to introduce hom-Lie algebroids. We would like to insist that it is not straightforward at all to see what this definition should be. This should not come as a surprise: a definitive notion of Hom-group, allowing to state Lie I, II and III theorems, is still to be found. In particular, there is no such a thing as a hom-Lie groupoid that could give us a hint. To derive a definition that makes sense, we indeed had to go through the notion of hom-Gerstenhaber algebra, but even there there was an unexpected phenomenon, a hom-Gerstenhaber algebra is not hom-associative, as one could have expected, hence defining a hom-Lie algebroid does not reduce simply adding the prefix hom- to classical definitions and results in a systematic manner.

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2. HOM-LIE ALGEBRAS AND HOM-POISSON ALGEBRAS

Given \mathfrak{g} a vector space and a bilinear map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, we call **automorphism of $(\mathfrak{g}, [\cdot, \cdot])$** a linear map $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$\alpha([x, y]) = [\alpha(x), \alpha(y)]$$

for all $x, y \in \mathfrak{g}$.

Definition 2.1. [HLS06] A **hom-Lie algebra** is a triple $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ with \mathfrak{g} a vector space equipped with a skew-symmetric bilinear map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ and an automorphism α of $(\mathfrak{g}, [\cdot, \cdot])$ such that:

$$(1) \quad [\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g} \quad (\text{hom-Jacobi identity}).$$

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A **morphism** between hom-Lie algebras $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha)$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \beta)$ is a linear map $\psi : \mathfrak{g} \rightarrow \mathfrak{h}$ such that $\psi([x, y]_{\mathfrak{g}}) = [\psi(x), \psi(y)]_{\mathfrak{h}}$ and $\psi(\alpha(x)) = \beta(\psi(x))$ for all $x, y \in \mathfrak{g}$. When \mathfrak{h} is a vector subspace of \mathfrak{g} and ψ is the inclusion map, one speaks of **hom-Lie subalgebra**.

In a similar fashion, one defines **graded hom-Lie algebras** to be triples $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ with $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ a graded vector space, $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ a graded skew-symmetric bilinear map of degree 0 and $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ an automorphism of $(\mathfrak{g}, [\cdot, \cdot])$ of degree 0 satisfying for all $x \in \mathfrak{g}_i, y \in \mathfrak{g}_j, z \in \mathfrak{g}_k$:

$$(-1)^{ik} [\alpha(x), [y, z]] + (-1)^{ji} [\alpha(y), [z, x]] + (-1)^{kj} [\alpha(z), [x, y]] = 0, \quad (\text{graded hom-Jacobi identity}).$$

Of course, these definitions make sense for finite dimensional or infinite dimensional vector spaces indifferently.

Example 2.2. (See e.g. [MS1]). Given a vector space \mathfrak{g} equipped with a skew-symmetric bilinear map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ and an automorphism $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ of $(\mathfrak{g}, [\cdot, \cdot])$, define $[\cdot, \cdot]_{\alpha} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$[x, y]_{\alpha} = \alpha([x, y]), \quad \forall x, y \in \mathfrak{g}.$$

Then $(\mathfrak{g}, [\cdot, \cdot]_{\alpha}, \alpha)$ is a hom-Lie algebra if and only if the restriction of $[\cdot, \cdot]$ to the image of α^2 is a Lie bracket. In particular, hom-Lie structures are naturally associated to Lie algebras equipped with a Lie algebra automorphism [Yau1]. Such hom-Lie structures are said to be **obtained by composition**.

Definition 2.3. [MS1] A **hom-associative algebra** is a triple (A, μ, α) consisting of a vector space A , a bilinear map $\mu : A \otimes A \rightarrow A$ and an automorphism α of (A, μ) satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)), \quad \forall x, y, z \in A \quad (\text{hom-associativity}).$$

Example 2.4. [Yau1] As in example 2.2, given (A, μ) an associative algebra and $\alpha : A \rightarrow A$ an algebra automorphism, the triple $(A, \mu_{\alpha} := \alpha \circ \mu, \alpha)$ is a hom-associative algebra, said again to be **obtained by composition**.

As one can expect, the commutator of a hom-associative algebra is a hom-Lie algebra:

Example 2.5. [MS1] For every hom-associative algebra (A, μ, α) (see definition 2.3 above), the triple $(A, [\cdot, \cdot], \alpha)$ is a hom-Lie algebra, where

$$[x, y] := \mu(x, y) - \mu(y, x)$$

for all $x, y \in A$.

A Poisson algebra being a space endowed with an associative and a Lie product, satisfying some compatibility relation, the next definition is perfectly natural:

Definition 2.6. [MS3] A **hom-Poisson algebra** is a quadruple $(A, \mu, \{\cdot, \cdot\}, \alpha)$ consisting of a vector space A , bilinear maps $\mu : A \otimes A \rightarrow A$ and $\{\cdot, \cdot\} : A \otimes A \rightarrow A$ and a linear map $\alpha : A \rightarrow A$ such that:

- (1) (A, μ, α) is a commutative hom-associative algebra,
- (2) $(A, \{\cdot, \cdot\}, \alpha)$ is a hom-Lie algebra,
- (3) $\{\alpha(x), \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\{x, y\}, \alpha(z))$, for all $x, y, z \in A$.

There is, however, a slightly related notion, that shall be useful in the sequel:

Definition 2.7. A **purely hom-Poisson algebra** is a quadruple $(A, \mu, \{\cdot, \cdot\}, \alpha)$ consisting of a vector space A , bilinear maps $\mu : A \otimes A \rightarrow A$ and $\{\cdot, \cdot\} : A \otimes A \rightarrow A$ and a linear map $\alpha : A \rightarrow A$ such that:

- (1) (A, μ) is a commutative associative algebra,
- (2) $(A, \{\cdot, \cdot\}, \alpha)$ is a hom-Lie algebra,
- (3) $\{x, \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\{x, y\}, \alpha(z))$, for all $x, y, z \in A$.

Example 2.8. [Yau2] Let $(A, \mu, \{\cdot, \cdot\})$ be a Poisson algebra and $\alpha : A \rightarrow A$ a Poisson automorphism, then the quadruple $(A, \mu_{\alpha} := \alpha \circ \mu, \{\cdot, \cdot\}_{\alpha} := \alpha \circ \{\cdot, \cdot\}, \alpha)$ is a hom-Poisson algebra, said to be **obtained by composition**. It is indeed enough to assume that $\{\cdot, \cdot\}$ (resp. μ) is a Lie bracket (resp. an associative product) when restricted to the image of α^2 .

Also, $(A, \mu, \{\cdot, \cdot\}_{\alpha} := \alpha \circ \{\cdot, \cdot\}, \alpha)$ is a purely hom-Poisson algebra.

Example 2.9. In particular, given (M, π) a manifold equipped with a bivector field π , and $\varphi : M \rightarrow M$ a smooth map, then a hom-Poisson structure on $C^\infty(M)$ can be obtained by composition provided that φ preserves the bivector field π (i.e. $\pi_{\varphi(m)} = (\wedge^2 T_m \varphi)(\pi_m)$ for all $m \in M$) and that the Schouten-Nijenhuis bracket $[\pi, \pi]$ is a trivector field that vanishes on $\varphi^2(M) \subset M$. Under these conditions, $(C^\infty(M), \mu_{\varphi^*}, \{, \}_{\varphi^*} = \varphi^* \circ \{, \}, \varphi^*)$ is a hom-Poisson algebra where μ is the usual product on $C^\infty(M)$ (and $(C^\infty(M), \mu, \{, \}_{\varphi^*}, \varphi^*)$ is a purely hom-Poisson algebra).

Example 2.9 makes the following definition natural: a triple (M, π, φ) , with π a bivector field on a manifold M and $\varphi : M \rightarrow M$ a smooth map, is called a **hom-Poisson manifold** when φ preserves the bivector field π and that the Schouten-Nijenhuis bracket and $[\pi, \pi]$ vanishes on $\varphi^2(M) \subset M$.

Example 2.10. Here are examples of hom-Poisson algebras that are not obtained by composition in general, see [BEM] for an alternative description. Let $(\mathfrak{g}, [,], \alpha)$ be a hom-Lie algebra. Equip its symmetric algebra $S(\mathfrak{g})$ with the product $\mu_\alpha := \alpha \circ \mu$, where $\mu(x, y) = x \odot y$ is the symmetric product and $\alpha : S(\mathfrak{g}) \rightarrow S(\mathfrak{g})$ stands for the automorphism of $(S(\mathfrak{g}), \mu)$ given by $\alpha(x_1 \odot \dots \odot x_n) := \alpha(x_1) \odot \dots \odot \alpha(x_n)$ for all $x_1, \dots, x_n \in \mathfrak{g}$. The quadruple $(S(\mathfrak{g}), \mu_\alpha, \{, \}_{\alpha}, \alpha)$ is a hom-Poisson algebra where

$$\{x_1 \odot \dots \odot x_p, y_1 \odot \dots \odot y_q\} := \sum_{i=1}^p \sum_{j=1}^q [x_i, y_j] \odot \alpha(x_1 \odot \dots \odot \widehat{x}_i \odot \dots \odot x_p \odot y_1 \odot \dots \odot \widehat{y}_j \odot \dots \odot y_q),$$

for all $x_1, \dots, x_p, y_1, \dots, y_q \in \mathfrak{g}$. Identifying $S(\mathfrak{g})$ with polynomial functions on \mathfrak{g}^* , we could also write:

$$\{F, G\}(a) = \left\langle \left[dF|_{\alpha^*(a)}, dG|_{\alpha^*(a)} \right], a \right\rangle$$

for all polynomial functions F, G on \mathfrak{g}^* and all $a \in \mathfrak{g}^*$, with the understanding that the differential of a function of \mathfrak{g}^* , a priori an element in $T^*\mathfrak{g}^*$ is considered as an element in \mathfrak{g} . It deserves to be noticed that \mathfrak{g}^* is *not* a hom-Poisson manifold in general. Also, it is not clear how we can associate a purely hom-Poisson algebra structure on $S(\mathfrak{g})$.

3. HOM-GERSTENHABER ALGEBRAS

Lie algebroids structures on a vector bundle $A \rightarrow M$ are in one-to-one correspondence with Gerstenhaber algebra structures on $\Gamma(\wedge^\bullet A[-1])$, see e.g. [KSM1, KSM2, McKX], making natural the idea of defining hom-Lie algebroids through the following object:

Definition 3.1. A **hom-Gerstenhaber algebra** is a quadruple $(\mathcal{A} = \oplus_{i \in \mathbb{Z}} \mathcal{A}_i, \wedge, [,], \alpha)$ where $(\mathcal{A} = \oplus_{i \in \mathbb{Z}} \mathcal{A}_i, \wedge)$ is a graded commutative associative algebra, α is an automorphism of (\mathcal{A}, \wedge) of degree 0 and $[,] : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is a bilinear map of degree -1 such that:

- (1) $(\mathcal{A}[1], [,], \alpha)$ is a graded hom-Lie algebra (as usual, $\mathcal{A}[1]$ refers to the graded vector space whose component of degree i is \mathcal{A}_{i+1} , for all $i \in \mathbb{Z}$);
- (2) the **hom-Leibniz rule** holds:

$$[X, Y \wedge Z] = [X, Y] \wedge \alpha(Z) + (-1)^{(i-1)j} \alpha(Y) \wedge [X, Z],$$

for all $X \in \mathcal{A}_i, Y \in \mathcal{A}_j, Z \in \mathcal{A}$.

Remark 3.2. Notice that \wedge is assumed to be an associative product, not a hom-associative product, so that a hom-Gerstenhaber algebra is not an odd version of a hom-Poisson algebra. But it might be seen as an odd version of a purely hom-Poisson algebra.

Example 3.3. Given an automorphism α of a Gerstenhaber algebra $(\mathcal{A}, \wedge, [,], \alpha)$ (i.e. α is an automorphism for both pairs (\mathcal{A}, \wedge) and $(\mathcal{A}, [,], \alpha)$), then $(\mathcal{A}, \wedge, \alpha \circ [,], \alpha)$ is a hom-Gerstenhaber algebra, said to be **obtained by composition**. Again, it suffices to assume that $[,]$ satisfies the Jacobi identity on the image of α^2 .

Example 3.4. Let \mathfrak{g} be a vector space equipped with a skew-symmetric bilinear map $[,] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ and an automorphism α of $(\mathfrak{g}, [,], \alpha)$. The triple $(\mathfrak{g}, [,], \alpha)$ is a hom-Lie algebra if and only if the quadruple

$(\wedge^\bullet \mathfrak{g}, \wedge, \llbracket, \rrbracket, \alpha)$ is a hom-Gerstenhaber algebra where

$$\llbracket x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_q \rrbracket = \sum_{i=1}^p \sum_{j=1}^q (-1)^{i+j} [x_i, y_j] \wedge \alpha(x_1 \wedge \cdots \widehat{x_i} \wedge \cdots \wedge x_p \wedge y_1 \wedge \cdots \widehat{y_j} \wedge \cdots \wedge y_q)$$

for all $x_1, \dots, x_p, y_1, \dots, y_q \in \mathfrak{g}$ and

$$\alpha(x_1 \wedge \cdots \wedge x_p) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_p).$$

The only difficult part is to prove that the graded hom-Jacobi identity holds on $\wedge^\bullet \mathfrak{g}$ if and only if \llbracket, \rrbracket satisfies the hom-Jacobi identity. This follows from the fact (which follows from a cumbersome but direct computation) that the hom-Jacobiator, defined as

$$Jac_\alpha(X, Y, Z) := (-1)^{(i-1)(k-1)} \llbracket \alpha(X), \llbracket Y, Z \rrbracket \rrbracket + (-1)^{(j-1)(i-1)} \llbracket \alpha(Y), \llbracket Z, X \rrbracket \rrbracket + (-1)^{(k-1)(j-1)} \llbracket \alpha(Z), \llbracket X, Y \rrbracket \rrbracket$$

for all $X \in \wedge^i \mathfrak{g}, Y \in \wedge^j \mathfrak{g}, Z \in \wedge^k \mathfrak{g}$, satisfies

$$(2) \quad Jac_\alpha(XY, Z, T) = \alpha^2(X) Jac_\alpha(Y, Z, T) + (-1)^{ij} \alpha^2(Y) Jac_\alpha(X, Z, T)$$

and is a graded skew-symmetric map, so that it vanishes if and only if its restriction to $\wedge^0 \mathfrak{g} = \mathbb{R}$ and $\wedge^1 \mathfrak{g} = \mathfrak{g}$ vanishes.

Let $(\mathfrak{g}, \llbracket, \rrbracket, \alpha)$ be a hom-Lie algebra, denote by α^s the s -power of α , $s \geq 1$, i.e.

$$\alpha^s = \alpha \circ \cdots \circ \alpha \quad (s \text{ times}).$$

For any element x in the hom-Lie algebra $(\mathfrak{g}, \llbracket, \rrbracket, \alpha)$ define the α^s -adjoint map $\text{ad}_x^s : \mathfrak{g} \rightarrow \mathfrak{g}$ by $\text{ad}_x^s y = [\alpha^s(x), y]$. Using the hom-Jacobi identity (1) we obtain

$$(3) \quad \text{ad}_{[x, y]}^s \circ \alpha = \text{ad}_{\alpha(x)}^s \circ \text{ad}_y^s - \text{ad}_{\alpha(y)}^s \circ \text{ad}_x^s.$$

Let us recall the following definition:

Definition 3.5. [Sheng] A **representation** of a hom-Lie algebra $(\mathfrak{g}, \llbracket, \rrbracket, \alpha)$ on a vector space V is a pair (ρ, α_V) of linear maps $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, $\alpha_V : V \rightarrow V$ such that:

$$(4) \quad \rho(\alpha(x)) \circ \alpha_V = \alpha_V \circ \rho(x)$$

$$(5) \quad \rho([x, y]) \circ \alpha_V = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x),$$

for all $x, y \in \mathfrak{g}$.

Examples 3.6. a. [Sheng] The α^s -adjoint map defines a representation (ad^s, α) of a hom-Lie algebra $(\mathfrak{g}, \llbracket, \rrbracket, \alpha)$ on the vector space \mathfrak{g} .

b. Let (ρ, α_V) be a representation of the hom-Lie algebra $(\mathfrak{g}, \llbracket, \rrbracket, \alpha)$ on the vector space V . Then $(\wedge^\bullet \mathfrak{g} \otimes S^\bullet(V), \wedge, \llbracket, \rrbracket, \alpha)$ is a hom-Gerstenhaber algebra where $\alpha : \wedge^\bullet \mathfrak{g} \otimes S^\bullet(V) \rightarrow \wedge^\bullet \mathfrak{g} \otimes S^\bullet(V)$ is defined as $\alpha(x_1 \wedge \cdots \wedge x_p \otimes v_1 \odot \cdots \odot v_q) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_p) \otimes \alpha_V(v_1) \odot \cdots \odot \alpha_V(v_q)$, the bracket \llbracket, \rrbracket being the hom-Gerstenhaber bracket whose restriction to $\wedge^0 \mathfrak{g} \otimes S(V)$ vanishes, whose restriction to $\wedge^\bullet \mathfrak{g} \otimes S^0(V) \simeq \wedge^\bullet \mathfrak{g}$ is as in example 3.4, and such that:

$$\llbracket x, v_1 \odot \cdots \odot v_q \rrbracket = \sum_{i=1}^q \rho(x)(v_i) \odot \alpha(v_1 \odot \cdots \odot \widehat{v_i} \odot \cdots \odot v_q),$$

for all $x \in \mathfrak{g}$ and $v_1, \dots, v_q \in V$.

Definition 3.7. Given an algebra automorphism α of a commutative associative algebra A , we call α -**derivation** a map $\delta : A \rightarrow A$ which satisfies

$$\delta(FG) = \alpha(F)\delta(G) + \alpha(G)\delta(F)$$

for all $F, G \in A$.

We denote by $\text{der}_\alpha(A)$ the set of all α -derivations.

Remark 3.8. Let M be a manifold and $\varphi : M \rightarrow M$ a smooth map. Then, $\text{der}_{\varphi^*}(C^\infty(M))$ can be identified with $\Gamma(\varphi^! TM)$, by mapping a section $X \in \Gamma(\varphi^! TM)$ to the φ^* -derivation mapping a function F to the function whose value at $m \in M$ is $X_m d_{\varphi(m)} F$, with the understanding that X_m must be considered as an element in $T_{\varphi(m)} M$.

Proposition 3.9. *For every hom-Gerstenhaber algebra $(\mathcal{A} = \oplus_{i \in \mathbb{N}} \mathcal{A}_i, \wedge, \llbracket, \rrbracket, \alpha)$, denote by $\rho : \mathcal{A}_1 \rightarrow \text{End}(\mathcal{A}_0)$ the map given by $\rho(X)[F] := \llbracket X, F \rrbracket$ for all $F \in \mathcal{A}_0$. Then \mathcal{A}_0 is a commutative associative algebra, $\alpha|_{\mathcal{A}_0}$ is an algebra automorphism of \mathcal{A}_0 , $F \mapsto \rho(X)(F)$ is, for all $X \in \mathcal{A}_1$, a $\alpha|_{\mathcal{A}_0}$ -derivation of \mathcal{A}_0 , the triple $(\mathcal{A}_1, \llbracket, \rrbracket|_{\mathcal{A}_1 \times \mathcal{A}_1}, \alpha|_{\mathcal{A}_1})$ is a hom-Lie algebra, and $(\rho, \alpha|_{\mathcal{A}_0})$ is a representation of $(\mathcal{A}_1, \llbracket, \rrbracket|_{\mathcal{A}_1 \times \mathcal{A}_1}, \alpha|_{\mathcal{A}_1})$ on \mathcal{A}_0 .*

Proof. Only the last point needs justification. We recover relations (4) and (5) by using that α is an automorphism of \llbracket, \rrbracket of degree 0 and the graded hom-Jacobi identity as follows,

$$\begin{aligned} \alpha|_{\mathcal{A}_0}(\rho(X)[F]) &= \rho(\alpha|_{\mathcal{A}_1}(X))[\alpha|_{\mathcal{A}_0}(F)] \\ \rho(\llbracket X, Y \rrbracket)[\alpha|_{\mathcal{A}_0}(F)] &= (\rho(\alpha|_{\mathcal{A}_1}(X)) \circ \rho(Y) - \rho(\alpha|_{\mathcal{A}_1}(Y)) \circ \rho(X))[F], \end{aligned}$$

for all $X, Y \in \mathcal{A}_1$ and $F \in \mathcal{A}_0$. \square

4. DEFINITION OF HOM-LIE ALGEBROID

We can now, at last, define hom-Lie algebroids:

Definition 4.1. A **hom-Lie algebroid** is a quintuple $(A \rightarrow M, \varphi, [\cdot, \cdot], \rho, \alpha)$, where $A \rightarrow M$ is a vector bundle over a manifold M , $\varphi : M \rightarrow M$ is a smooth map, $[\cdot, \cdot] : \Gamma(A) \otimes \Gamma(A) \rightarrow \Gamma(A)$ is a bilinear map, called **bracket**, $\rho : \varphi^!A \rightarrow \varphi^!TM$ is a vector bundle morphism, called **anchor**, and $\alpha : \Gamma(A) \rightarrow \Gamma(A)$ is a linear endomorphism of $\Gamma(A)$ such that

1. $\alpha(FX) = \varphi^*(F)\alpha(X)$, for all $X \in \Gamma(A), F \in C^\infty(M)$;
2. the triple $(\Gamma(A), [\cdot, \cdot], \alpha)$ is a hom-Lie algebra;
3. the following hom-Leibniz identity holds:

$$[X, FY] = \varphi^*(F)[X, Y] + \rho(X)[F]\alpha(Y), \quad \forall X, Y \in \Gamma(A), F \in C^\infty(M).$$

4. (ρ, φ^*) is a representation of $(\Gamma(A), [\cdot, \cdot], \alpha)$ on $C^\infty(M)$.

Remarks 4.2. **a.** Linear endomorphisms of $\Gamma(A)$, $\alpha : \Gamma(A) \rightarrow \Gamma(A)$, satisfying $\alpha(FX) = \varphi^*(F)\alpha(X)$ for all $X \in \Gamma(A), F \in C^\infty(M)$ are in one-to-one correspondence with vector bundle morphisms from $\varphi^!A$ to A over the identity of M . Given $X \in \Gamma(A)$, a section of the pull-back bundle $\varphi^!A$ is given by mapping $m \in M$ to $X_{\varphi(m)} \in A_{\varphi(m)} \simeq (\varphi^!A)_m$. Applying a vector bundle morphism from $\varphi^!A$ to A over the identity of M to that section yields a section of A , and the henceforth defined assignment α satisfies $\alpha(FX) = \varphi^*(F)\alpha(X)$, for all $X \in \Gamma(A), F \in C^\infty(M)$. Moreover, every endomorphism of $\Gamma(A)$ satisfying this relation is of that form.

b. The hom-Leibniz identity implies that, given sections X, Y of A , the value of $[X, Y]$ at a given point $m \in M$ depends only on the first jet of X and Y at $\varphi(m)$.

c. Above, $\rho(X)[F]$ stands for the function on M whose value at $m \in M$ is $\langle d_{\varphi(m)}F, \rho_m(X_{\varphi(m)}) \rangle$ where $\rho_m : (\varphi^!A)_m \simeq A_{\varphi(m)} \rightarrow (\varphi^!TM)_m \simeq T_{\varphi(m)}M$ is the anchor map evaluated at $m \in M$ and $X_{\varphi(m)}$ is the value of the section $X \in \Gamma(A)$ at $\varphi(m) \in M$.

Example 4.3. When α (hence φ) is the identity map, a hom-Lie algebroid $(A \rightarrow M, \varphi, \alpha, [\cdot, \cdot], \rho)$ is simply a Lie algebroid [McK]. A hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is a hom-Lie algebroid over a singleton. More generally, define an **action of a hom-Lie algebra** $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ on the manifold M , equipped with a smooth map $\varphi : M \rightarrow M$, to be a linear map δ from \mathfrak{g} to the space of φ^* -derivations such that (δ, φ^*) defines a representation of the hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ on the vector space $C^\infty(M)$. Then a hom-Lie algebroid is obtained by considering the trivial vector bundle $A = M \times \mathfrak{g} \rightarrow M$, the linear map α_A mapping $Fc_v \rightarrow \varphi^*(F)c_{\alpha(v)}$ for all $v \in \mathfrak{g}, F \in C^\infty(M)$, the anchor ρ mapping $A_{\varphi(m)} \simeq \mathfrak{g}$ to the element of $T_{\varphi(m)}M$ given by the pointwise derivation $F \mapsto \delta(v)[F]|_m$ and the bracket given by:

$$[Fc_v, Gc_w] = \varphi^*(FG)c_{[v, w]} + \varphi^*(F)\rho(v)[G]c_{\alpha(w)} - \varphi^*(G)\rho(w)[F]c_{\alpha(v)}$$

for all $F, G \in C^\infty(M), v, w \in \mathfrak{g}$. In the previous, c_v, c_w denote the constant sections of $M \times \mathfrak{g} \rightarrow M$ given by $m \mapsto (v, m)$ and $m \mapsto (w, m)$ respectively. This hom-Lie algebroid is not obtained by composition in general.

The following theorem is a consequence of proposition 3.9, and will allow us to give more examples.

Theorem 4.4. *Let $A \rightarrow M$ be a vector bundle, $\varphi : M \rightarrow M$ a smooth map, $\alpha : \Gamma(A) \rightarrow \Gamma(A)$ a linear endomorphism satisfying $\alpha(FX) = \varphi^*(F)\alpha(X)$, for all $X \in \Gamma(A), F \in C^\infty(M)$. Denote by α again its extension to $\alpha : \Gamma(\wedge^\bullet A) \rightarrow \Gamma(\wedge^\bullet A)$ given by:*

$$(6) \quad \alpha(FX_1 \wedge \cdots \wedge X_p) = \varphi^*(F)\alpha(X_1) \wedge \cdots \wedge \alpha(X_p)$$

for all $p \in \mathbb{N}$, $X_1, \dots, X_p \in \Gamma(A)$, $F \in C^\infty(M)$. Then there is a one-to-one correspondence between hom-Gerstenhaber algebra structures on $(\Gamma(\wedge^\bullet A), \wedge, \llbracket, \rrbracket, \alpha)$ and hom-Lie algebroids structures on $(A \rightarrow M, \varphi, [\cdot, \cdot], \rho, \alpha)$, obtained as follows:

- (1) Given a hom-Gerstenhaber algebra structure $(\Gamma(\wedge^\bullet A), \wedge, \llbracket, \rrbracket, \alpha)$, we define a bracket $[\cdot, \cdot]$ on $\Gamma(A)$ by restriction of \llbracket, \rrbracket to $\Gamma(A)$ and an anchor $\rho : \varphi^!A \rightarrow \varphi^!TM$ by $\rho(X)[F] := \llbracket X, F \rrbracket$ for all $X \in \Gamma(A), F \in C^\infty(M)$.
- (2) Conversely, given a hom-Lie algebroid structure $(A \rightarrow M, \varphi, [\cdot, \cdot], \rho, \alpha)$, we define a hom-Gerstenhaber bracket on $\Gamma(\wedge^\bullet A)$, for all $X_1, \dots, X_p, Y_1, \dots, Y_q \in \Gamma(A)$, $q \geq 1$, $F \in C^\infty(M)$, by:

$$\llbracket X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q \rrbracket = \sum_{i=1}^p \sum_{j=1}^q (-1)^{i+j} [X_i, Y_j] \wedge \alpha(X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge \widehat{Y}_j \wedge \cdots \wedge Y_q)$$

and by

$$\llbracket X_1 \wedge \cdots \wedge X_p, F \rrbracket = \sum_{i=1}^p (-1)^{i+1} \rho(X_i)[F] \wedge \alpha(X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_p).$$

Proof. 1) We first need to justify our definition of ρ . It follows from the hom-Leibniz identity of the hom-Gerstenhaber algebra that $F \mapsto \llbracket X, F \rrbracket$ is a φ^* -derivation and that $\llbracket GX, F \rrbracket = \varphi^*(G)\llbracket X, F \rrbracket$ for all $X \in \Gamma(A), F, G \in C^\infty(M)$. Altogether, these properties imply that there is a unique vector bundle morphism $\rho : \varphi^!A \rightarrow \varphi^!TM$ such that $\rho(X)[F] = \llbracket X, F \rrbracket$, for all $X \in \Gamma(A), F \in C^\infty(M)$. Condition 1 in definition 4.1 holds by assumption. Conditions 2 and 4 follow from proposition 3.9. The hom-Leibniz identity of the hom-Gerstenhaber algebra:

$$\llbracket X, FY \rrbracket = \varphi^*F\llbracket X, Y \rrbracket + \llbracket X, F \rrbracket \alpha(Y) \quad \forall X, Y \in \Gamma(A), F \in C^\infty(M)$$

gives the condition 3 and proves the first item, since $\llbracket X, F \rrbracket = \rho(X)[F]$ by the very construction of ρ .

2) The hom-Leibniz identity, together with the facts that $\rho(FX)[G] = \varphi^*(F)\rho(X)[G]$ for all $F, G \in C^\infty(M), X \in \Gamma(A)$ and that $F \mapsto \rho(X)[F]$ is a φ^* -derivation imply that \llbracket, \rrbracket is well-defined. It also implies that it obeys to the hom-Leibniz identity. By equation (2), it suffices to check the hom-Jacobi identity of \llbracket, \rrbracket for triples made of three sections of $\Gamma(A)$ and triples made of two sections of A together with one function on M . In the first case, it simply follows from the hom-Jacobi identity of $[\cdot, \cdot]$ and in the second case, it is equivalent to the assumption that (ρ, φ^*) is a representation of $(\Gamma(A), [\cdot, \cdot], \alpha)$. This proves 2).

The constructions of both items are clearly inverse one to the other, and the theorem follows. \square

The theorem shall in fact help us to construct examples of hom-Lie algebroids.

Example 4.5. Recall [McK] that a Gerstenhaber algebra structure $(\Gamma(\wedge^\bullet A), \llbracket, \rrbracket, \wedge)$ is naturally associated to every Lie algebroid $(A, [\cdot, \cdot], \rho)$. Given $\alpha : \Gamma(A) \rightarrow \Gamma(A)$ a linear endomorphism of $\Gamma(A)$ satisfying $\alpha(FX) = \varphi^*(F)\alpha(X)$, for all $X \in \Gamma(A), F \in C^\infty(M)$ ($\varphi : M \rightarrow M$ being a given smooth map), an algebra endomorphism, again called α , of $(\Gamma(\wedge^\bullet A), \wedge)$ can be constructed as in (6). This morphism α preserves the bracket \llbracket, \rrbracket provided that $\alpha : \Gamma(A) \rightarrow \Gamma(A)$ is a Lie algebra morphism such that $\rho(\alpha(X))[\varphi^*F] = \varphi^*(\rho(X)[F])$ for all $X \in \Gamma(A), F \in C^\infty(M)$. Example 3.3 then allows us to build a hom-Gerstenhaber algebra by composition, which, by theorem 4.4, yields a hom-Lie algebroid. Indeed, it suffices that \llbracket, \rrbracket satisfies the Jacobi identity on the image of α^2 . For this, it suffices that \llbracket, \rrbracket satisfies the Jacobi identity when applied to triples of the form $(\alpha^2(X), \alpha^2(Y), \alpha^2(Z))$ or $(\alpha^2(X), \alpha^2(Y), (\varphi^*)^2(F))$, with $X, Y, Z \in \Gamma(A), F \in C^\infty(M)$. This means that $(A, [\cdot, \cdot], \rho)$ can be just assumed to be a pre-Lie algebroid (i.e. the Leibniz rule holds, but $[\cdot, \cdot]$ is not a Lie bracket) that satisfies the Jacobi identity on the image of $\alpha^2 : \Gamma(A) \rightarrow \Gamma(A)$ and such that for all $X, Y \in \Gamma(A)$:

$$(7) \quad \rho([\alpha^2(X), \alpha^2(Y)]) - [\rho(\alpha^2(X)), \rho(\alpha^2(Y))]$$

is a vector field on M that vanishes on the image of $\varphi^2 : M \rightarrow M$.

We now describe two particular cases of the previous construction.

Example 4.6. A vector field $\mathcal{V} \in \mathfrak{X}(M)$ induces a Lie algebroid as follows: $A_{\mathcal{V}} := M \times \mathbb{R}$ is the trivial line bundle, the Lie bracket of two sections $F, G \in C^\infty(M) = \Gamma(A_{\mathcal{V}})$ is given by $[F, G] := F\mathcal{V}[G] - G\mathcal{V}[F]$ and the anchor of $F \in C^\infty(M) = \Gamma(A_{\mathcal{V}})$ is the vector field $F\mathcal{V}$. Let $\varphi : M \rightarrow M$ be a smooth map preserving \mathcal{V} , i.e. $\varphi^*(\mathcal{V}[F]) = \mathcal{V}[\varphi^*F]$. Then φ^* is a linear endomorphism of $C^\infty(M) = \Gamma(A_{\mathcal{V}})$ which satisfies the required conditions to yield a hom-Lie algebroid by composition.

Example 4.7. Let (M, π, φ) be a hom-Poisson manifold. Let $A = T^*M$, and let α be the pull-back morphism $\varphi^* : \Gamma(T^*M) \rightarrow \Gamma(T^*M)$. Then a pre-Lie algebroid structure [Vaisman] is defined on $\Gamma(T^*M)$ by considering the anchor map $\rho := \pi^\# : T^*M \rightarrow TM$ together with the bracket

$$[a, b]_\pi := \mathcal{L}_{\pi^\#a}b - \mathcal{L}_{\pi^\#b}a + d_\pi(a \wedge b).$$

It is immediate that $\varphi^* : \Gamma(T^*M) \rightarrow \Gamma(T^*M)$ is an automorphism of this bracket and that $\rho(\varphi^*a)[\varphi^*F] = \varphi^*(\rho(a)[F])$ for any 1-form $a \in \Gamma(T^*M)$ and $F \in C^\infty(M)$. Assuming the Schouten-Nijenhuis bracket $[\pi, \pi]$ to vanish on the image of φ^2 amounts to require that $[\cdot, \cdot]_\pi$ satisfies the Jacobi identity when restricted to the image of $(\varphi^*)^2 : \Gamma(T^*M) \rightarrow \Gamma(T^*M)$ and that condition (7) holds. A hom-Lie algebroid can therefore be constructed by composition.

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